Proof by induction that

$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for all integers $n \geq 1$.

Induction hypothesis: The general case of the induction hypothesis, $IH(n)$, is

$$f(n) = \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Basis step: $IH(1)$ predicts that

$$f(1) = \sum_{i=0}^{1} i^2 = \frac{1(1+1)(2 \cdot 1+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1$$

This is the sum of the squares from 0 to 1, so $IH(1)$ is true.

Induction step:

Expectation: $IH(n+1)$ is

$$f(n+1) = \sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$$

Connecting cases for n and $n+1$: we can observe that

$$\begin{aligned} f(n+1) = \sum_{i=0}^{n+1} i^2 &= 1^2 + 2^2 + 3^3 + \ldots + n^2 + (n+1)^2 \\ &= \sum_{i=0}^{n} i^2 + (n+1)^2 \\ &= f(n) + (n+1)^2 \end{aligned}$$

Assuming $IH(n)$ is true, then

$$\begin{aligned} f(n+1) &= f(n) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{2n^3 + 9n^2 + 13n + 6}{6} \\ &= \frac{(n+1)(n+2)(2(n+1)+1)}{6} \end{aligned}$$

This matches the expectation, proving that if $IH(n)$ is true, then $IH(n+1)$ must also be true.